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P. P. BANERJEE,
W. CHOE, G. CAO
and
W. Hereman

UNIVERSITY OF WISCONSIN

CENTER FOR THE MATHEMATICAL SCIENCES

Center for the Mathematical Sciences
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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STATIONARY EIGENMODES AND THEIR STABILITY DURING WAVE PROPAGATION IN A MEDIUM WITH QUADRATIC AND CUBIC NONLINEARITIES WITHOUT DISPERSION

P. P. Banerjee[1,*], W. Choe[1], G Cao[1,2] and W. Hereman[3,**]

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ABSTRACT

Stationary eigenmodes are derived for wave propagation in a medium with quadratic and cubic nonlinearities. Aperiodic algebraic solitary waves are the eigenmodes with a continuous spectrum while periodic solitary waves are those with a spectrum comprising an infinite number of harmonics. Stationary eigenmodes comprising the fundamental and the second harmonic only have also been derived. The stability of the aperiodic solitary waves have been studied numerically and a stabilization technique proposed. The stability of stationary eigenmodes for the two-frequency case is also discussed.

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[1]Department of Electrical Engineering, Syracuse University, Syracuse, NY 13244.

[2]Present address: Department of Physiology, Medical College of Virginia, Virginia Commonwealth University, Richmond, VA 23298.

[3]Also Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706.

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1. Introduction

It is well known that solitary waves can exist in a medium with a quadratic nonlinearity and dispersion. The physical basis for solitary wave formation in such a system is the critical balance between the steepening effect of nonlinearity and the smoothening effect of dispersion. Since solitary waves do not change their shape during propagation, they can be termed stationary traveling wave eigenmodes (with a continuous spectrum) of the system [1]. [Unless otherwise stated, we will use the term "solitary waves" to denote aperiodic (e.g. pulse-type) solutions.] If, furthermore, the solitary waves preserve their shape upon mutual interaction, they may be called "solitons" [2]. Periodic solitary waves (e.g. cnoidal waves) have a discrete spectrum which, however contains a fundamental frequency and an infinite number of harmonics.

Analogous to the continuous spectrum case, stationary eigenmodes may also exist for the case of a discrete number of propagating frequencies. As shown in [3], stationary eigenmodes comprising frequencies ω_0 and $2\omega_0$ may exist in a medium with quadratic nonlinearity and dispersion; and perturbations around the steady state exhibit periodic or "recurrent" behavior [4], similar to the celebrated Fermi-Pasta-Ulam (FPU) recurrence [5].

In this paper, we look at the possibility of stationary eigenmodes (with continuous as well as discrete spectra) in a medium with quadratic and cubic nonlinearities, and <u>no</u> dispersion. A physical picture for the existence of solitary waves may be portrayed as follows: Consider, for instance, a kinematic wave equation of the form [6]

$$\frac{\partial\psi}{\partial t} + c_0\ (1 + \beta_2\psi + \beta_3\psi^2)\ \frac{\partial\psi}{\partial x} = 0, \qquad (1)$$

where ψ represents the wavefunction, c_0 is the (linear) phase velocity and β_2, β_3 denote the quadratic and cubic nonlinearity coefficients. Consider, for now, the case where $\beta_2 > 0$, $\beta_3 < 0$ and where ψ at $t = 0$ is a baseband (pulse-type) signal greater than zero. Then with time, the leading edge of the pulse steepens while the trailing edge smoothens under the action of the quadratic nonlinearity alone, while the reverse occurs under the effect of the cubic nonlinearity (see Fig. 1). The combined effect can be visualized as a balancing process whereby the signal may finally evolve into a shape which remains unchanged during propagation.

The organization of the paper is as follows: In Section 2, we discuss aperiodic and periodic solitary wave solutions of a model equation with quadratic and cubic nonlinearities only, viz., the nonlinear Klein-Gordon equation without dispersion. Results for one-dimensional propagation only are presented; for extension to higher dimensions, the reader is referred to [7]. In Section 3, numerical simulations on interactions of aperiodic solitary wave solutions are presented. Results show instability upon collision; however, this can be removed by incorporating a small amount of "saturating" fourth-order nonlinearity in the system. In Section 4, we study the two frequency case, viz., the fundamental and its second harmonic, by starting from the kinematic wave equation (1) and establish relations between the spectral amplitudes in the stationary case. A stability analysis of the stationary eigenmode in the discrete frequency case is also performed; these results are presented in Section 5. Perturbations around the steady state sometimes exhibit periodic behavior, similar to FPU recurrence. For the solution to the initial value problem, the reader is referred to [8].

2. Aperiodic and Periodic Solitary Wave Solutions of the Nonlinear Klein Gordon Equation without Dispersion

We shall take, as our model equation, the nonlinear Klein-Gordon (NKG) equation without dispersion in the form

$$\frac{\partial^2 \psi}{\partial t^2} - c_0{}^2 \frac{\partial^2 \psi}{\partial x^2} = A_2 \psi^2 + A_3 \psi^3; \ \psi = \psi\ (x,t), \tag{2}$$

where c_0, as in (1), represents the linear phase velocity and where A_2, A_3 denote the quadratic and cubic nonlinearity coefficients. In our quest for solitary wave solutions to (2), we introduce a traveling frame of reference and renormalize according to

$$\zeta = \frac{1}{\sqrt{c_0^2 - v^2}} \left(\frac{A_2^2}{A_3}\right)^{1/2} (x - vt), \ A_3 > 0,$$

$$\Psi = - \frac{A_3}{A_2} \psi\ , \ \Psi = \Psi(\zeta) \tag{3}$$

to get

$$\frac{d^2 \Psi}{d\zeta^2} = \Psi^2 - \Psi^3. \tag{4}$$

Now multiplying (4) by $d\Psi/d\zeta$ and integrating w.r.t. ζ, we obtain

$$\left(\frac{d\Psi}{d\zeta}\right)^2 = \frac{2}{3} \Psi^3 - \frac{1}{2} \Psi^4 + K\ , \tag{5}$$

where K is an integration constant. Assuming that at $\zeta = 0$, $\Psi = \Psi_0$ and $d\Psi/d\zeta = 0$, we get

$$K = \frac{1}{2} \Psi_0{}^4 - \frac{2}{3} \Psi_0{}^3. \tag{6}$$

In order to reduce (5) to a tractable integral, we put

$$\Psi = \hat{\Psi} + \alpha \tag{7}$$

and set the constant term equal to zero; this gives

$$\left(\frac{d\hat{\Psi}}{d\zeta}\right)^2 = (2\alpha^2 - 2\alpha^3)\ \hat{\Psi} + (2\alpha - 3\alpha^2)\hat{\Psi}^2 + (\frac{2}{3} - 2\alpha)\ \hat{\Psi}^3 - \frac{1}{2} \hat{\Psi}^4 \tag{8}$$

with either

$$\alpha = \alpha_1 = \Psi_0, \tag{9}$$

or given by the (real) roots of the cubic equation

$$\alpha^3 + (\Psi_0 - \frac{4}{3})\ \alpha^2 + (\Psi_0 - \frac{4}{3})\Psi_0\ \alpha + (\Psi_0 - \frac{4}{3})\ \Psi_0^2 = 0. \tag{10}$$

In what follows, we will discuss the two cases (a) $\Psi_0 = \frac{4}{3}$ and

(b) $\Psi_0 = \frac{4}{3}$ one by one. It turns out that case (a) yields the aperiodic (algebraic) solution while case (b) gives periodic solutions.

2.1 Case (a): $\Psi_0 = 4/3$

The possible values of α are $\alpha_1 = 4/3$ (from (9)) and $\alpha_2 = 0$ (from (10)). We will discuss the latter possibility; it can be easily shown that the other choice leads to identical results.

For the latter choice, K = 0 and the ODE in (5) can be readily solved to give the aperiodic algebraic solitary wave of the form (see Fig. 2)

$$\Psi(\zeta) = \frac{4/3}{1+2\zeta^2/9} . \qquad (11)$$

Note the similarity of (11) with the solitary wave solution of the Benjamin-Ono equation [9]. Note that the amplitude of the solitary wave is fixed through the choice of the nonlinear coefficients A_2, A_3 (see Eq.(3)). The velocity $v(<c_0)$ may be related to the width of the solitary wave through use of the same equation.

2.2 Case (b): $\Psi_0 \neq 4/3$

The possible values of α to examine are now given by (9) and the roots of the cubic equation (10). We state here a lemma, without proof, regarding the roots of the cubic equation. The proof of this is detailed in [7].

Lemma 2.1 Of the three roots of (10), one is real and the other two are complex conjugates.

Thus, it suffices to consider the two real roots of $\alpha(=\alpha_{1,2})$ for each value of Ψ_0. Now, from (8), upon writing

$$\hat{\Psi} = \frac{1}{\phi} , \tag{12}$$

we have

$$\left(\frac{d\phi}{d\zeta}\right)^2 = a\phi^3 + b\phi^2 + c\phi + d \overset{\Delta}{=} p(\phi), \tag{13}$$

where

$$a = 2\alpha^2-2\alpha^3 \ ; \ b = 2\alpha-3\alpha^2 \ ; \ c = \frac{2}{3}-2\alpha \ ; \ d = -\ 1/2. \tag{14}$$

The solution ϕ, and hence $\hat{\Psi}$ and Ψ, can thus be expressed in terms of Jacobian elliptic functions [10]. Hence Ψ is periodic in nature. Examining (4), it is obvious that if $d\Psi(0)/d\zeta = 0$, then $\Psi_0 > 1(<1)$ corresponds to the maximum (minimum) value of Ψ. We shall call the maximum (minimum) value $\Psi_{M(m)}$ with Ψ_{ext} denoting either of the two.

Now, it may be readily seen that Ψ_{ext} and α both satisfy a quartic equation of the form as in Eq. (6), with Ψ_0 replaced by Ψ_{ext} and α respectively. Since we have shown that α has two real roots, they must be Ψ_M and Ψ_m. Thus if $\Psi_{M(m)}$ is the given initial value, $\Psi_{m(M)}$ will follow as the solution to (10). Furthermore, straightforward arguments may be advanced to show that if $1 < \Psi_M < 4/3$, then $0 < \Psi_m < 1$, and if $\Psi_M > 4/3$, then $\Psi_m < 0$.

In order to explicitly evaluate the solution to (13), we have to calculate the roots of the polynomial $p(\phi)$. We state, again, a lemma without proof concerning the roots of the cubic polynomial. The proof is detailed in [7].

<u>Lemma 2.2</u> The polynomial $p(\phi)$ in (13) has only one real root

$$\beta = \frac{1}{\Psi_{ext}-\alpha} \tag{15}$$

where $\alpha = \Psi_{M(m)}$ if $\Psi_{ext} = \Psi_{m(M)}$.

The above lemma exposes four different cases to be considered. Corresponding to a given initial condition $\Psi_0 = \Psi_{ext}$ $(=\Psi_{M(m)})$, there are two values of α; $\alpha_1 > 1$ and $\alpha_2 < 1$. Suppose the initial condition is Ψ_M. The two values of α are Ψ_M and Ψ_m . If, next, we choose the initial condition as the Ψ_m corresponding to the Ψ_M chosen above, we will, once again, get two values of α, which are the same as before. Rigorous relationships between these four solutions have been derived in [7]; in this writeup we will only present selected results. This is in view of the fact that the second order ODE as in (4) should have a unique solution once two initial conditions (viz., the value of the function and its derivative at $\zeta = 0$) have been specified, irrespective of the value of α.

For instance, for $\alpha = \alpha_1 = \Psi_M (>1)$, we obtain from (13),

$$\pm \sqrt{-a_1} \int_{\zeta}^{\zeta_1} d\zeta = \int_{\phi}^{\beta_1} \frac{d\phi}{\sqrt{-p_1(\phi)}} \tag{16}$$

where

$$\beta_1 = \beta\Big|_{\alpha=\alpha_1=\Psi_M} = \frac{1}{\Psi_{ext} - \alpha}\Big|_{\alpha=\alpha_1=\Psi_M} = \frac{1}{\Psi_m - \Psi_M} \quad \text{(using (15))}, \tag{17}$$

$$\bar{p}_1(\phi) = \frac{1}{a_1} p(\phi)\Big|_{\alpha=\alpha_1=\Psi_M} = \phi^3 + \frac{b_1}{a_1}\phi^2 + \frac{c_1}{a_1}\phi + \frac{d_1}{a_1} \quad \text{(using (13))}, \tag{18}$$

where a_1, b_1, c_1, d_1 are defined by (14) with $\alpha = \alpha_1 = \Psi_M$, and ζ_1 denotes an integration constant to be determined from the initial condition(s). After some algebra and using (7), (12) and (14) it follows that [10]

$$\Psi_1(\zeta) = \alpha_1 + \frac{1 + cn[(2\alpha_1^3-2\alpha_1^2)^{1/2} \lambda_1(\zeta-\zeta_1)]}{(\beta_1-\lambda_1^2)+(\beta_1+\lambda_1^2)cn[(2\alpha_1^3-2\alpha_1^2)^{1/2} \lambda_1(\zeta-\zeta_1)]} \tag{19}$$

where

$$\lambda_1^2 = [\bar{p}_1'(\beta_1)]^{1/2}, \tag{20}$$

and where the parameter $\cos^2\delta_1$ of the cn function is given by

$$\cos^2 \delta_1 = 1 - m_1 = \frac{1}{2} + \frac{1}{8} \bar{p}_1''(\beta_1)/[\bar{p}_1'(\beta_1)]^{1/2} . \tag{21}$$

The period of the cn function, and hence of Ψ_1, is given as [10]

$$\Lambda_1 = \frac{4\bar{K}(1-m_1)}{(2\alpha_1^3 - \alpha_1^2)^{1/2} \lambda_1} \tag{22}$$

where

$$\bar{K}(\mu) \triangleq \int_0^{\pi/2} (1-\mu \sin^2\theta)^{-1/2} d\theta. \tag{23}$$

Corresponding to initial conditions Ψ_M and Ψ_m, the respective constants ζ_1 in (19) may be calculated. The results are, using (17),

$$\Psi_{1M(m)}(\zeta) = \Psi_M + \frac{1 \mp cn[(2\Psi_M^3-2\Psi_M^2)^{1/2}\lambda_1\zeta]}{(\beta_1-\lambda_1^2) \mp (\beta_1+\lambda_1^2) cn[(2\Psi_M^3-2\Psi_M^2)^{1/2}\lambda_1\zeta]}. \tag{24}$$

It is easily seen that there is a half-period shift $(=2\bar{K}(1-m_1))$ between Ψ_{1M} and Ψ_{1m}, as is to be expected.

As an example, consider the case where $\Psi_0 = 2$; $\Psi'(0) = 0$. From (24), the analytic solution reads

$$\Psi_M(\zeta) = \frac{1 + 4.947\ cn(1.530\zeta)}{3.041 - 0.067\ cn(1.530\zeta)} \tag{25}$$

with $1-m_1 = 0.571$ and $\Lambda = 5.015$. This is plotted in Fig. 3.

In passing, we remark that whether the periodic solution can be decomposed into a series of associated aperiodic solitary waves, as can be achieved for solutions of the KdV equation [11], is still under investigation.

3. Interaction of Algebraic Solitary Waves and their Stability

In this Section, we present a numerical scheme to test the stability of the algebraic solitary waves derived in (11) upon mutual interaction and propose a method to "stabilize" them. The procedure followed is similar to a recipe from nonlinear optics where radially symmetric solutions in a medium with cubic nonlinearity are sometimes stabilized by incorporating a small amount of a saturable fifth-order nonlinearity in the system [12].

The numerical scheme essentially employs an explicit finite-difference method [13] to discretize the NKG equation without dispersion (Eq. (2)) as:

$$\psi_{j+1}^{i} = c_0^2(\Delta t/\Delta x)^2\ (\psi_j^{i+1} + \psi_j^{i-1} - 2\psi_j^i) - (\psi_{j-1}^{i} - 2\psi_j^i)$$

$$+ (\Delta t)^2\ [A_2(\psi_j^i)^2 + A_3(\psi_j^i)^3], \tag{26}$$

where i and j refer to the number of increments, Δx and Δt, in space and time respectively. However, for numerical analysis, it is required to show that the above scheme is convergent. Unfortunately, the proof of convergence is rather involved and, even when possible, a satisfactory theory can only be advanced for linear PDEs. We follow here a suggestion by Ablowitz et al [14] that a certain degree of stabilization can be reached in the numerical scheme by using the average of several adjacent space increments to replace ψ_j^i in the nonlinear term(s), viz,

$$(\psi_j^i)^n \longrightarrow (\tfrac{1}{3}\ (\psi_j^{i-1} + \psi_j^i + \psi_j^{i+1}))^n, \tag{27}$$

and by using $\Delta x = c_0\Delta t$.

Fig. 4 shows the three regions we employ for our computer simulations. I and II are linear nondispersive while III represents the quadratically and cubically nonlinear region. Our numerical experiment involves making two (identical) pulses traveling to the right and left in regions I and II respectively. These pulses eventually enter region III where they interact with each other. Outflow (matched) boundary conditions are simulated at x = 0 and x = 8 (see Fig. 4) to prevent any reflections from the extremities. Interface boundary conditions at x = 2 and x = 6 are set such that the wavefunction and its spatial derivative are continuous across each interface.

The initial conditions are set from a knowledge of the single aperiodic solitary wave solution. Note that from (11), the denormalized algebraic solitary wave may be written, using (3), as

$$\psi(x,t) = \frac{-\dfrac{4A_2}{3A_3}}{1 + \dfrac{2A_2^2\ (x-vt)^2}{9A_3(c_0^2-v^2)}}\ . \tag{28}$$

As one initial condition we thus set

$$\psi(x,0) = 2 \times (-4/3)(A_2/A_3) \left\{ \frac{1}{1 + \dfrac{2A_2^2 \; (x-x_1)^2}{9A_3(c_0^2-v^2)}} + \frac{1}{1 + \dfrac{2A_2^2 \; (x-x_2)^2}{9A_3(c_0^2-v^2)}} \right\} \tag{29}$$

where x_1, x_2 lie in regions I and II respectively. The differential initial condition viz., $\psi_t(x,0)$, is set equal to zero. This facilitates propagation of each pulse in both positive and negative directions, starting from each linear region. At $t = 0$, the right and left propagating pulses overlap, accounting for double the amplitude $2\left(-\dfrac{4A_2}{3A_3}\right)$. After some time, each initial pulse breaks up into two with amplitudes equal to $\left(-\dfrac{4}{3}\right)\left(\dfrac{A_2}{A_3}\right)$, one traveling toward region III, the other toward an extreme matched boundary (at $x = 0$ or $x = 8$).

We choose $v = 1.0$, $c_0 = 1.0050$ $[c_0^2 - v^2 = 0.01]$, $A_2 = -1$, $A_3 = 1$ and $x_1 = 1$, $x_2 = 6$ for our numerical experiment. The results are plotted in the sequence of Figs. 5(a)-(g), from which we may easily see that the initial waveforms distort severely due to interaction in the nonlinear region. No appreciable distortion is observed at the linear-nonlinear interface.

In what follows, we show that the observed distortion may be removed by incorporating a higher order saturating nonlinearity in the system. The equation describing the modified system can be expressed as

$$\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} = A_2\psi^2 + A_3\psi^3 + A_4\psi^4 \; ; \; A_2 < 0, \; A_3 > 0, \; A_4 < 0. \tag{30}$$

Eq. (30) is then programmed with the same values of A_2, A_3, v amd c_0 as before, and with $A_4 = -0.12$. This value of A_4 was chosen from a series of trial values incorporated to minimize distortion upon interaction. The results are plotted in the sequence of Figs. 6(a)-(g), showing that with the help of stabilization by the saturating fourth-order term, the algebraic solitary waves, again, preserve their shape upon interaction. Numerical experiments with two solitary waves moving in the same direction (but with different velocities, and hence, widths) and with sequences of algebraic solitary waves are currently under way.

4. Steady States in the Discrete Case: Fundamental and Second Harmonic

In the discussion above, aperiodic and periodic solitary wave solutions were derived for a medium with quadratic and cubic nonlinearities, using the NKG equation without dispersion as a model. The aperiodic solitary waves can be regarded as stationary traveling wave eigenmodes but having a continuous frequency spectrum. The periodic solitary waves, however, can be thought of as a collection of discrete frequencies (viz, the fundamental and its infinite set of harmonics) that form a stationary eigenmode. In this section, we will show that it is possible to establish stationary eigenmodes even for two frequencies, viz., the fundamental and its second harmonic, in a medium with quadratic and cubic nonlinearities. We will take, as our starting point, the nonlinear kinematic wave equation as in (1), which though different from the NKG equation (2), nevertheless provides a good model for such a medium.

To study the spatial evolution of the spectral amplitudes at the fundamental (ω_0) and the second harmonic ($2\omega_0$), we substitute

$$\psi(x,t) = \frac{1}{2}\sum_{n=-2}^{2} G_n(x)e^{jn(\omega_0 t-k_0 x)} \qquad (31)$$

in (1) with $G_0 = 0$ and $G_{-n} = G_n^*$ to ensure that the wavefunction is real. The ratio ω_0/k_0, where k_0 represents the propagation constant of the fundamental, is taken to be equal to the linear phase velocity c_0 in (1). In general, the propagation constant for the nth harmonic may differ from nk_0; however, if so, it should show up in the expression for $G_n(x)$. Using the "slowly-varying" assumption

$$\left|\frac{dG_n}{dx}\right| << \left|nk_0 G_n\right|, \tag{32}$$

together with the normalizations

$$ue^{-j\phi_1} \triangleq G_1/R, \qquad ve^{-j\phi_2} \triangleq G_2/R\ ,$$

$$R \triangleq (\left|G_1(0)\right|^2 + \left|G_2(0)\right|^2)^{1/2}\ ,$$

$$\xi = \frac{1}{2}\beta_2 R k_0 x, \tag{33}$$

and the definitions

$$Q \triangleq \frac{\beta_3 R}{2\beta_2},$$

$$\theta \triangleq \phi_2 - 2\phi_1, \tag{34}$$

we finally arrive at a set of three coupled equations linking the (real) spectral amplitudes u,v and the relative phase difference θ.

$$\frac{du}{dv} = uv \sin\theta. \tag{35a}$$

$$\frac{dv}{d\xi} = -u^2 \sin\theta, \tag{35b}$$

$$\frac{d\theta}{d\xi} = (2v - \frac{u^2}{v})\cos\theta + 2Q(v^2 - u^2)\ , \tag{35c}$$

with the relation

$$u^2 + v^2 = 1 \tag{36}$$

expressing the conservation of energy.

Eqns. (35) may be compared with the evolution equations derived by Armstrong et al [14] in connection with harmonic generation in optics in a (quadratically) nonlinear, dispersive dielectric. While Eqns. (35a,b) are identical to Armstrong's, in (35c) only the first term on the RHS is identical, while the second term is to be compared with Δs in [14] which represents the phase mismatch between the fundamental and the second harmonic in a dispersive medium.

The "relative phase-locked" stationary eigenmode may now be defined as follows:

$$\frac{du}{d\xi} = \frac{dv}{d\xi} = \frac{d\theta}{d\xi} = 0 \tag{37}$$

From (35c), this means (using (33) and (34), (36))

$$\left|G_{1s}\right| = \left(\frac{2 + \alpha|G_{2s}|}{1 + \alpha|G_{2s}|} \right)^{1/2} \left|G_{2s}\right| \quad ; \ \alpha \triangleq \frac{\beta_3}{\beta_2 \cos\theta} \tag{38}$$

where the subscript s has been used to denote the steady state. Observe from (35a,b) that possible values of θ_s in this case are 0 and π ($\cos\theta_s = \pm 1$). The variation of the second harmonic amplitude with the fundamental amplitude for the stationary eigenmode is plotted in Fig. 7. Note that when $\alpha < 0$, $|G_{2s}|$ is a multi-valued function of $|G_{1s}|$. The explicit relations for ϕ_1 and ϕ_2 may be obtained from the respective explicit differential equations (not written here) which occur as an intermediate step in the derivation of (35c). It may be readily checked that ϕ_1 and ϕ_2 vary linearly with ξ. Thus, the propagation constants of the fundamental and the second harmonic are indeed modified in the steady state. However, the new (constant) phase velocity is different from c_0.

In the following section, we will perform a stability analysis of the stationary eigenmode(s) to find which branches of the steady state plots (see Fig. 7) are stable and which are not. Whenever possible, the period of the perturbation will also be calculated.

For a description of the solution to the initial value problem starting from Eqns.(35), the reader referred to [8].

5. Stability Analysis of the Stationary Eigenmode (Fundamental and Second Har monic)

We first rewrite (35c) as

$$\frac{d(\sin\theta)}{d\xi} = (2v-\frac{u^2}{v})(1-\sin^2\theta) \pm 2Q(v^2-u^2)\sqrt{1-\sin^2\theta} \tag{39}$$

and perturb the system (35a,b), (39) as:

$$\begin{aligned} u &= u_s + \Delta u\,, \\ v &= v_s + \Delta v\,, \\ \sin\theta &= (\sin\theta)_s + \Delta(\sin\theta). \end{aligned} \tag{40}$$

Up to first order in the perturbations, this gives the following linearized set:

$$\begin{aligned} \frac{d(\Delta u)}{d\xi} &= u_s v_s\,\Delta(\sin\theta), \\ \frac{d(\Delta v)}{d\xi} &= -u_s^2\Delta(\sin\theta), \\ \frac{d[\Delta(\sin\theta)]}{d\xi} &= (3 + v_s^{-2} \pm 8Qv_s)\,\Delta v. \end{aligned} \tag{41}$$

Now assuming the explicit forms of the perturbations as proportional to $e^{\lambda\xi}$, the permissible values of λ (for nontrivial solutions) are:

$$\begin{aligned} \lambda_1 &= 0, \\ \lambda_{2,3} &= \pm j\sqrt{u_s^2(3 + v_s^{-2} \pm 8Qv_s)}. \end{aligned} \tag{42}$$

The condition for stability requires that the expression under the radical sign should be positive; in denormalized form this translates to

$$|G_{2s}| \gtrless - \frac{4R^2 - 3|G_{1s}|^2}{4\alpha(R^2 - |G_{1s}|^2)} \quad \text{for } \alpha \gtrless 0. \tag{43}$$

Consider the case $\alpha > 0$. It may be easily checked from (43) that any value of $|G_{1s}|$ and $|G_{2s}|$ (as given in Fig. 7(a)) will be stable. For $\alpha < 0$, the stable and unstable regions are drawn in Fig. 8. When this figure is juxtaposed on Fig. 7(b), it is clearly seen that the upper "branch" in Fig. 7(b) is unstable, while the lower "branch" is stable. We conclude, therefore, that the stationary eigenmodes in a medium with quadratic and cubic nonlinearities can have two stable states in the two-frequency (fundamental and second harmonic) case.

6. *Conclusion*

Stationary traveling wave eigenmodes, with continuous as well as discrete spectra, have been derived for a medium with quadratic and cubic nonlinearities, and no dispersion. The physical basis for the existence of a balance between the different nonlinear effects has been established. Aperiodic (algebraic) solitary waves, which occur as solutions to the NKG equation without dispersion, are an illustration of stationary eigenmodes with continuous spectra. These are, however, unstable upon mutual interaction, but may be "stabilized" by adding the right amount of a higher order saturating nonlinearity in the system. Periodic solitary waves in such a system may be expressed in terms of ratios of 'cn' functions, and may be looked upon as stationary eigenmodes with discrete, albeit infinitely many, frequency components. For yet another nondispersive system which is quadratically and cubically nonlinear, but modeled by a different equation, we have established stationary eigenmodes comprising the fundamental and the second harmonic and examined their stability. Not reported in this paper, but presented in [8] is the fact that in the solution to the initial value problem in this case, periodic exchange of energy between the fundamental and the second harmonic is observed, similar to FPU recurrence.

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References

1. D. W. McLaughlin and A. C. Scott, in Solitons in Action, eds. K. Lonngren and A.C. Scott, Academic, N.Y. 1978.
2. N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15, 240, 1965.
3. A. Korpel, J. Acoust. Soc. Am. 67, 1954, 1980.
4. P. P. Banerjee and A. Korpel, J. Acoust. Soc. Am. 70, 157, 1981.
5. E. Fermi, J. R. Pasta and S. M. Ulam, Los Alamos Sci. Lab. Rep. LA-1940, 1955.
6. G. B. Whitham, Linear and Nonlinear Waves, Wiley, N.Y. 1974.
7. P. P. Banerjee and G. Cao, J. Phys. A 21, 55, 1988.
8. W. Choe and P. P. Banerjee, "A frequency approach to wave propagation in a medium with quadratic and cubic nonlinearities", submitted to J. Phys. A.
9. H. Ono, J. Phys. Soc. Japan 39, 1082, 1975.
10. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, N.Y. 1970.
11. A. Korpel and P. P. Banerjee, Phys. Lett. 82A, 113, 1981.
12. V. E. Zakharov and V. S. Synakh, Sov. Phys. JETP 41, 465, 1976.
13. M. J. Ablowitz, M. D. Kruskal and J. F. Ladik, SIAM J. Appl. Math. 36, 428, 1979.
14. J. A. Armstrong, N. Bloembergen, J. Ducuing and P. S. Pershan, Phys. Rev. 127, 1918, 1962.

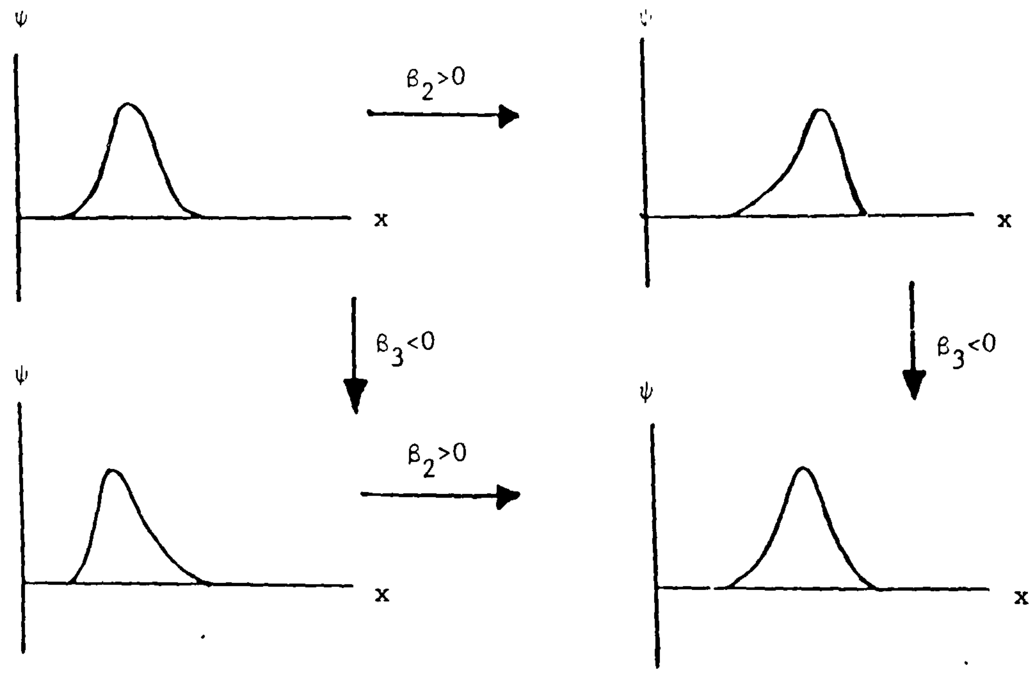

Fig.1 Plots showing balancing of quadratic nonlinearity
with a cubic nonlinearity.

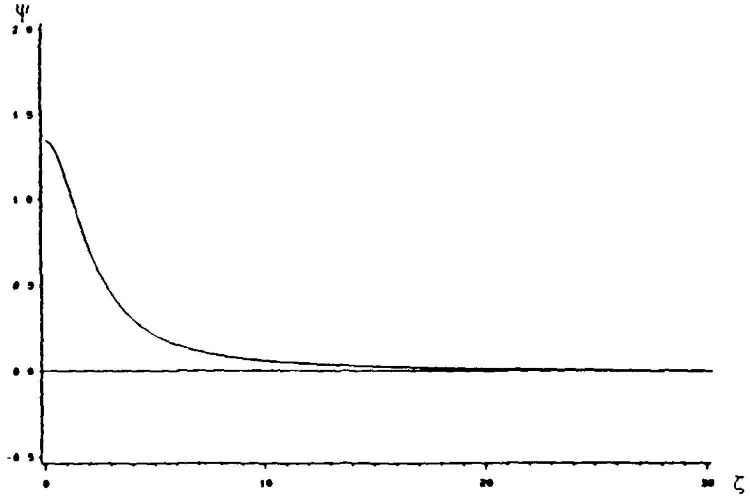

Fig.2 Profile of the aperiodic algebraic solitary wave

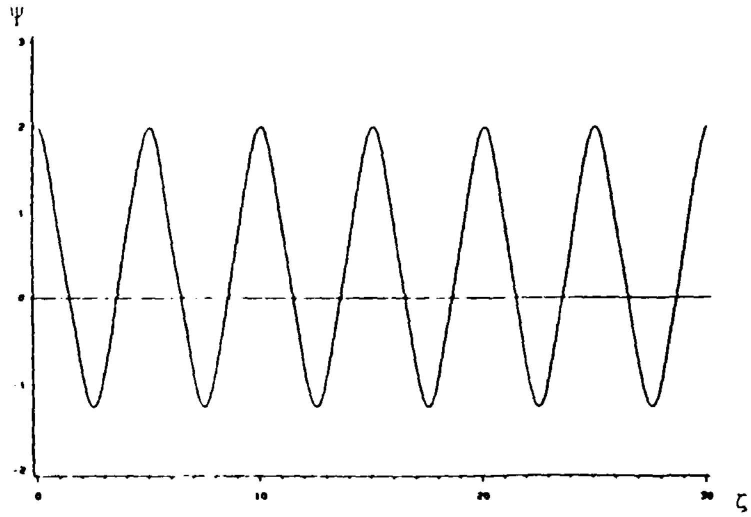

Fig.3 Profile of a periodic solitary wave with $\Psi_o=2$

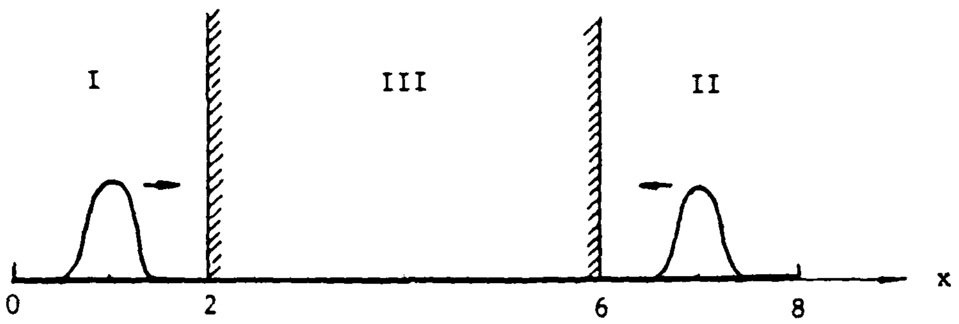

Fig.4 The three regions for the numerical experiment. Regions I and III are linear nondispersive; region II is nonlinear and dispersive.

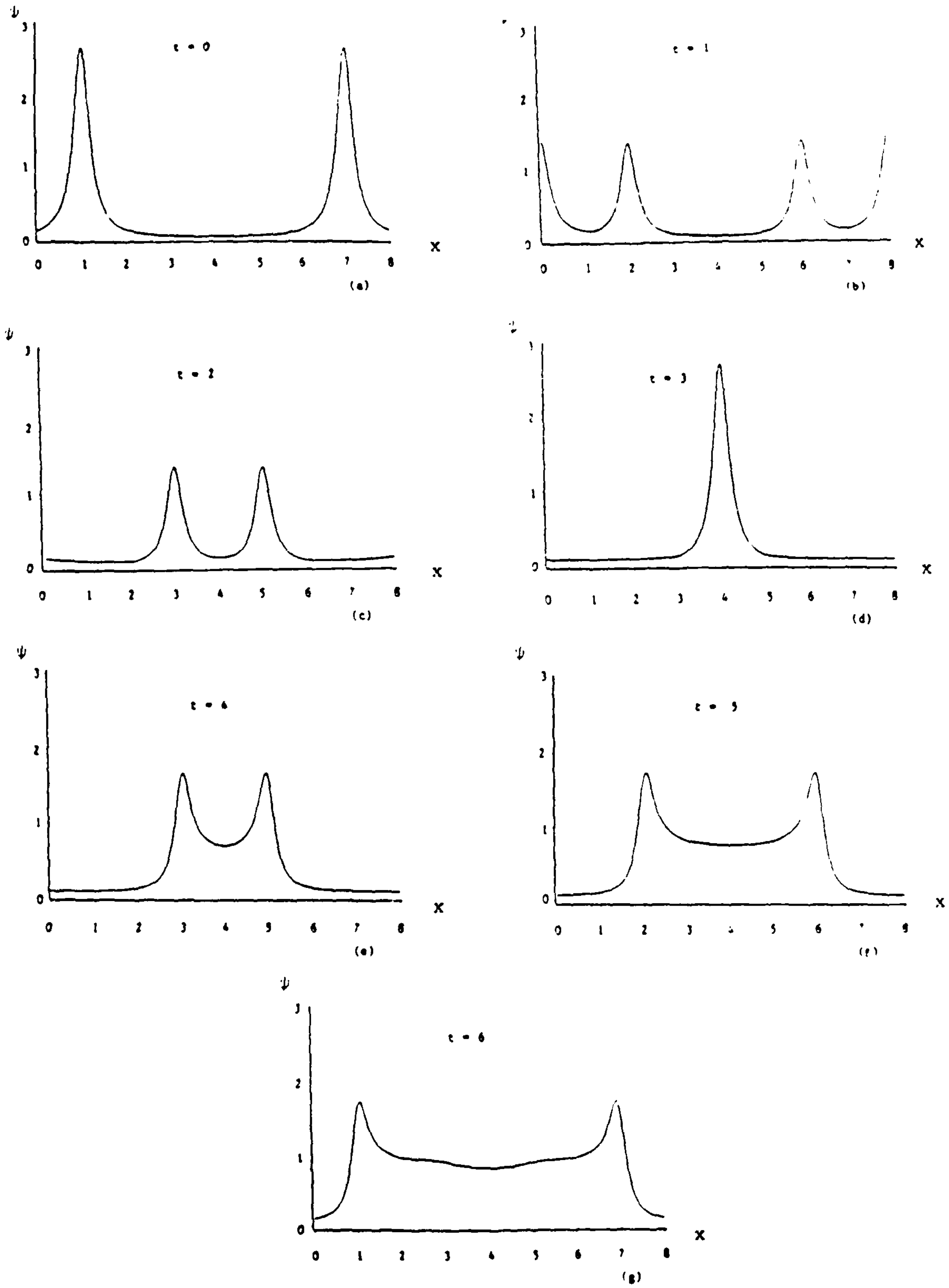

Fig.5 Interaction of algebraic solitary waves in a medium with quadratic and cubic nonlinearities.

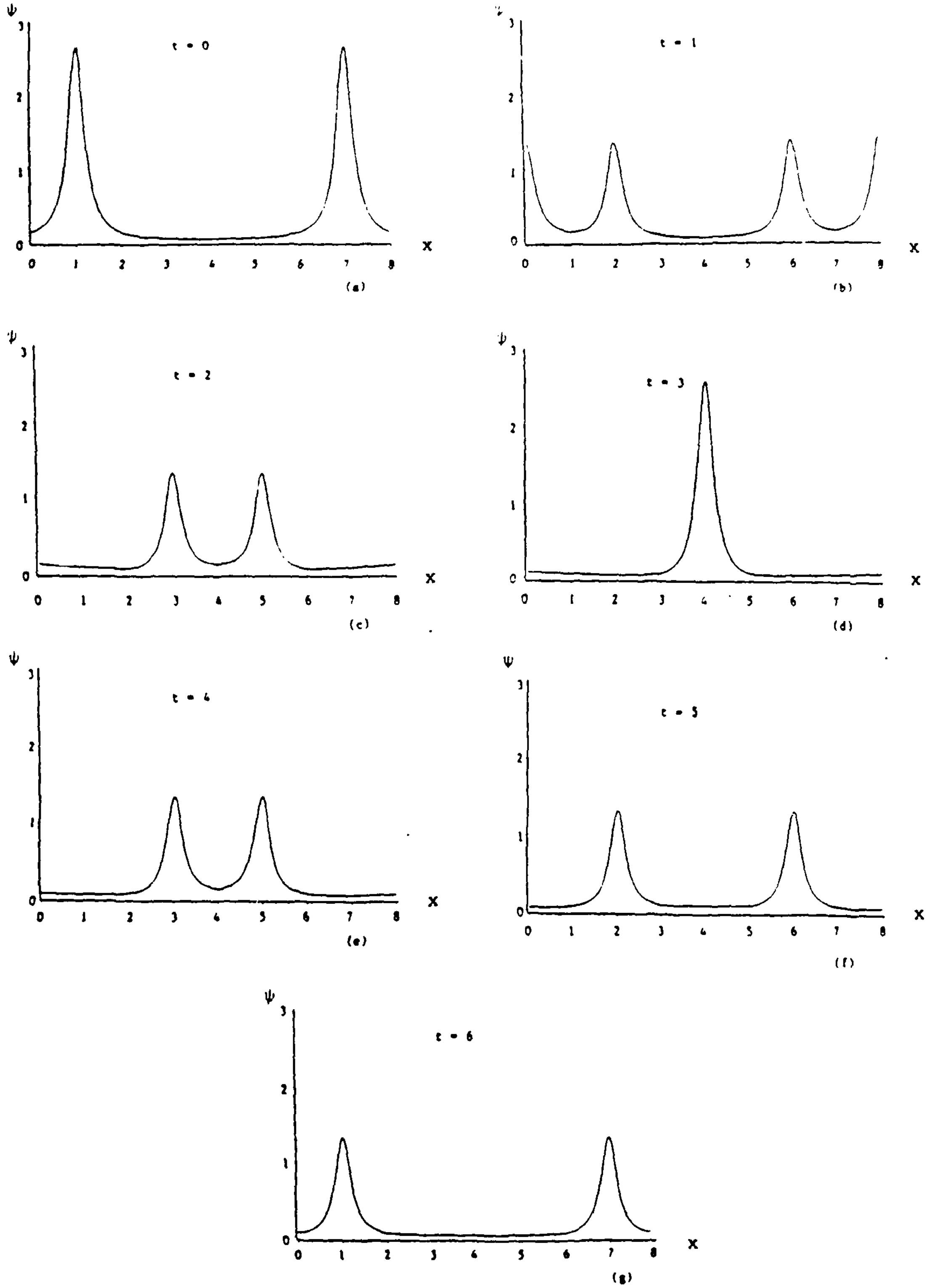

Fig.6 Stabilization of algebraic solitary waves upon interaction by incorporating a small amount of saturating fourth order nonlinearity.

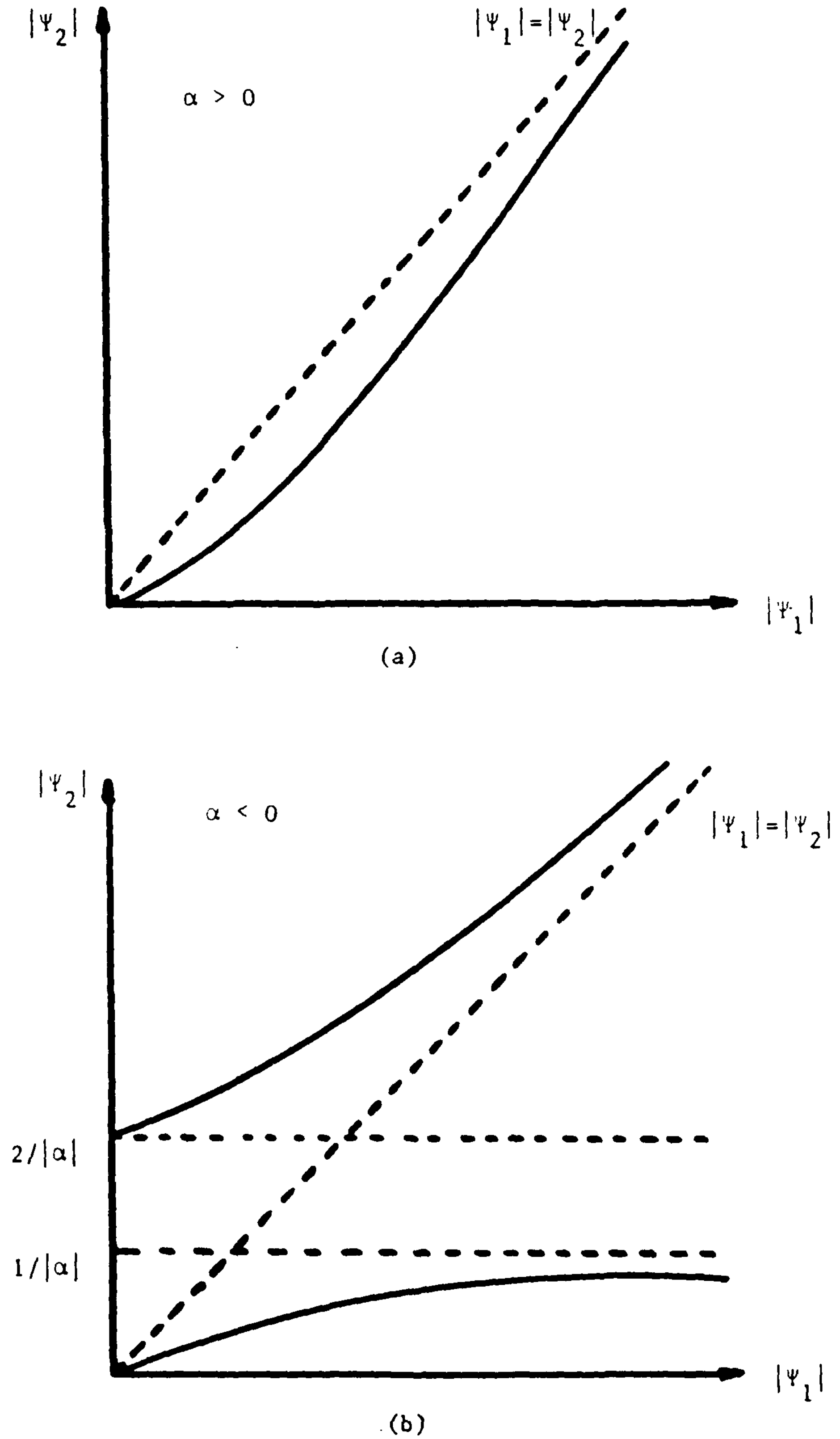

Fig.7 Plots of the second harmonic amplitude vs. the fundamental in the steady state.

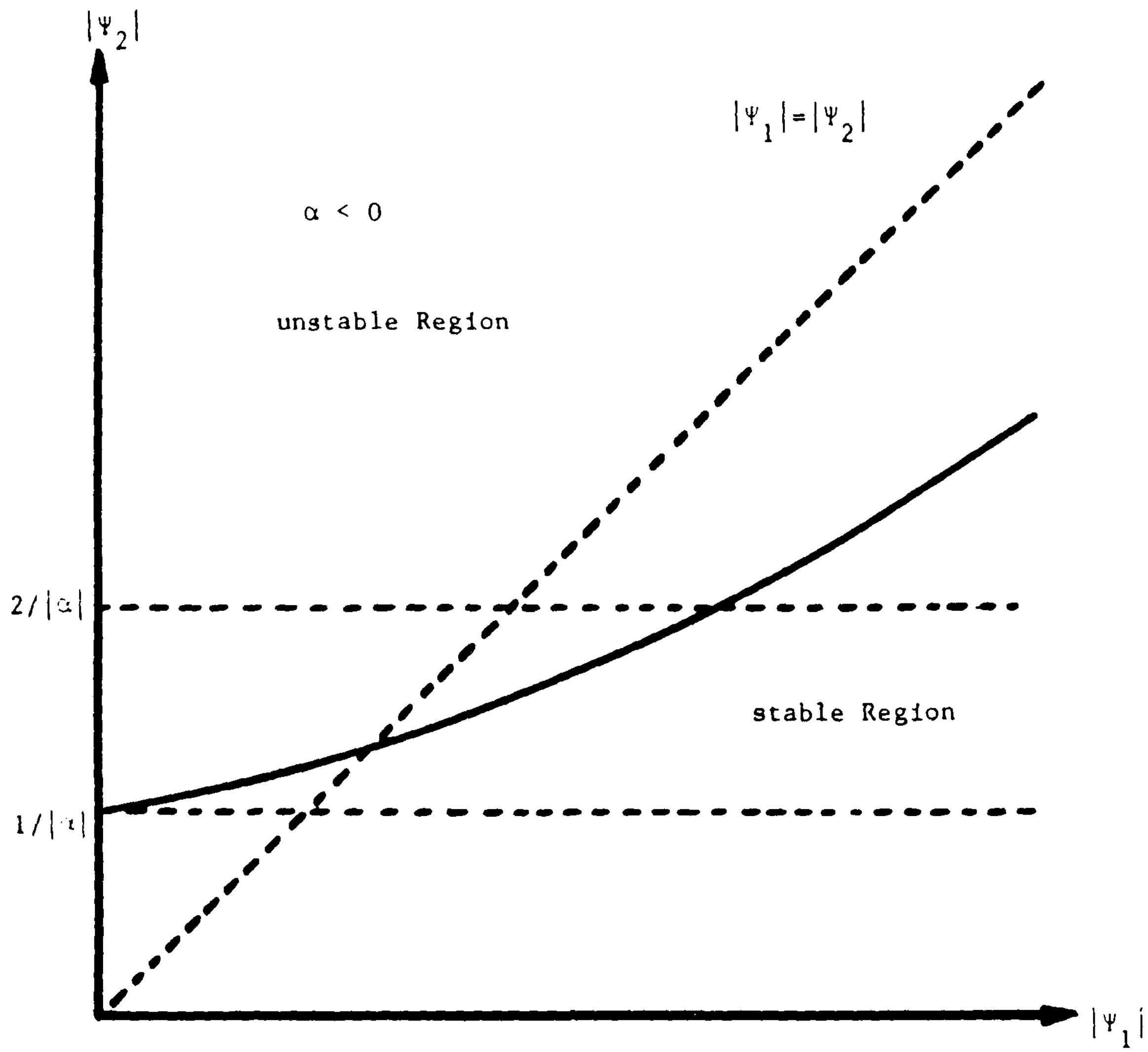

Fig.8 Stable and unstable regions around the steady state(s).